Reduction of β-integrable 2-Segre structures

Thomas Mettler

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Abstract

It is shown that over a $2n$-manifold M equipped with a β-integrable 2-Segre structure $\mathcal{S}$, there exists a quasiholomorphic fibre bundle $\rho : X_{\mathcal{S}} \to M$ with fibre $\mathbb{CP}^1 \setminus \mathbb{RP}^1$. We prove that ρ-sections having holomorphic image are in one-to-one correspondence with reductions of $\mathcal{S}$ to torsion-free $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$-structures on M. Consequently every β-integrable 2-Segre structure can locally be reduced to a torsion-free $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$-structure. In the homogeneous case on the oriented 2-plane Grassmannian $M = G_2^+(\mathbb{R}^{n+2})$, the reductions are in one-to-one correspondence with the smooth quadrics $Q \subset \mathbb{CP}^{n+1}$ without real points.

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1 Introduction

A subset $\mathcal{S}$ of a real vector space V is called an (m, n)-*Segre cone*, $m, n \geq 2$, if there exists a linear isomorphism $V \simeq \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^n)$ identifying $\mathcal{S}$ with the set of linear maps $\mathbb{R}^m \to \mathbb{R}^n$ of rank one. An (m, n)-*Segre structure* $\mathcal{S}$ on a smooth mn-manifold M is a smoothly varying choice of an (m, n)-Segre cone $\mathcal{S}_p \subset T_pM$ in each tangent space of M. In the case where $m + n$ is odd, an (m, n)-Segre structure is the same as a factoring of TM as the tensor product of two vector bundles over M of rank m and n respectively, together with an isomorphism of their determinant line bundles.

Segre structures were studied by Chern [9] from the viewpoint of projective geometry. Subsequently, Segre - or closely related structures and their counterparts in the category of complex manifolds were studied under various names, including tensor product structure [17, 18], Grassmannian spinor structure [21], generalised conformal structure [15], complex paraconformal structure [4], (almost) Grassmann structure [1, 18, 20] and in [5] as an example of a class of structures called almost symmetric hermitian manifolds.

(2,2)-Segre structures correspond to conformal structures of split signature (2,2) and thus provide examples of Segre structures on 4-manifolds. The Grassmannian $G_m(\mathbb{R}^{n+m})$ of m-planes in $\mathbb{R}^{n+m}$ carries a homogeneous (m, n)-Segre structure. In [16], it was shown that a path geometry satisfying a certain curvature condition induces a Segre structure on its space of paths. In particular the path geometry on the $(n+1)$-sphere whose paths are the "great circles" gives rise to a $(2, n)$-Segre structure on $G_2(\mathbb{R}^{n+2})$. Generalising the construction of [16], it was observed in [10] that every path geometry induces a Segre structure on an associated slit vector bundle and that this Segre structure descends to the path space of the path geometry if and only if the curvature condition of [16] is satisfied. In [13, 23], it was shown that every projective structure on a surface gives rise to a (anti) self-dual split-signature conformal 4-manifold admitting a conformal Killing vector field. Segre structures also arise in the study of multidimensional webs [3, 14].

In §2, after recalling some definitions, we associate a first order $G(m, n)$-structure $\pi : F_{\mathcal{S}} \to M$ to a given Segre structure $\mathcal{S}$ on a smooth manifold M. Here $G(m, n)$ is the Lie subgroup of $\mathrm{GL}((\mathbb{R}^m)^* \otimes \mathbb{R}^n) \subset \mathrm{GL}(mn, \mathbb{R})$ which leaves the Segre cone of decomposable vectors invariant. We restrict to the case $m = 2$ and simply speak of 2-Segre structures. Following ideas from [11, 24] for the construction of 'twistor spaces', we construct an almost complex structure $\mathfrak{J}$ on the total space of the fibre bundle $\rho : X_{\mathcal{S}} = F_{\mathcal{S}}/(S^1 \cdot \mathrm{GL}(n, \mathbb{R})) \to M$ from an $\mathcal{S}$-adapted linear connection on $F_{\mathcal{S}}$. The group $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$ is the subgroup of $\mathrm{GL}(n, \mathbb{C}) \subset \mathrm{GL}(2n, \mathbb{R})$ generated by $\mathrm{GL}(n, \mathbb{R})$ and the central subgroup $S^1 \subset \mathrm{GL}(n, \mathbb{C})$ consisting of scalar multiplication by unit complex numbers. We show that for a 2-Segre structure with $n > 2$, the almost complex structure $\mathfrak{J}$ on $X_{\mathcal{S}}$, associated to any torsion-free $\mathcal{S}$-adapted linear connection is integrable. For $n = 2$ it is integrable if in addition to torsion freeness a certain curvature condition is satisfied which corresponds to self-duality of the associated conformal 4-manifold. In the integrable case, the ρ-fibres are shown to be holomorphically embedded complex submanifolds. In this case we will say that $(X_{\mathcal{S}}, \mathfrak{J})$ is a *quasiholomorphic* fibre bundle. In particular, it follows with [2, 20] that for every so-called β-integrable 2-Segre structure $\mathcal{S}$, there exists a canonical almost complex structure $\mathfrak{J}$ on $X_{\mathcal{S}}$ so that $(X_{\mathcal{S}}, \mathfrak{J})$ is a quasiholomorphic fibre bundle.

In §3, we show that in the β-integrable case, ρ-sections having holomorphic image are in one-to-one correspondence with reductions of $\mathcal{S}$ to torsion-free $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$-structures on M. It follows that locally every β-integrable 2-Segre structure can be reduced to a torsion-free $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$-structure. Note that in [8], it was observed that $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$ does occur as

the holonomy group of torsion-free affine connections in dimension 2n.

In §4, we define the notion of 2-orientability of a 2-Segre structure. We show that the Grassmannian of oriented 2-planes $G_2^+(\mathbb{R}^{n+2})$, carries a 2-oriented 2-Segre structure together with an adapted torsion-free flat connection. In this canonical flat case on $M = G_2^+(\mathbb{R}^{n+2})$, the bundle $\rho : X_{\mathcal{S}} \to M$ is shown to be isomorphic to the bundle $\rho_0 : \mathbb{CP}^{n+1} \setminus \mathbb{RP}^{n+1} \to G_2^+(\mathbb{R}^{n+2})$, where the base point projection is given by $[z] \mapsto \Pi_{[z]} = \mathbb{R}\{\mathrm{Re}(z), \mathrm{Im}(z)\}$ and the 2-plane $\Pi_{[z]}$ is oriented by declaring $\mathrm{Re}(z), \mathrm{Im}(z)$ to be a positively oriented basis. Using a corollary from [22], it is shown that the torsion-free $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$-reductions on $G_2^+(\mathbb{R}^{n+2})$ are in one-to-one correspondence with the smooth quadrics $Q \subset \mathbb{CP}^{n+1}$ without real points.

Throughout the article smoothness, i.e. infinite differentiability is assumed and we adhere to the convention of summing over repeated indices.

Remark. In [8], Bryant observed that the space of oriented geodesics Λ of a geodesically simple Finsler structure of positive constant flag curvature (CFC) inherits a Kähler structure and a torsion-free $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$-structure satisfying a certain positivity condition. On October 13, 2009, Robert Bryant informed the author about his private notes regarding the generality of positive CFC Finsler structures on the n-sphere. He shows that a positive CFC Finsler structure on the n-sphere all of whose geodesics are closed and of the same length gives rise to a D^2-bundle $\rho : X \to \Lambda$ whose total space is a complex manifold. This bundle is isomorphic to $\rho_0 : \mathbb{CP}^{n+1} \setminus \mathbb{RP}^{n+1} \to G_2^+(\mathbb{R}^{n+2})$ in the case of a rectilinear Finsler structure. In addition, the Finsler structure induces a ρ-section having holomorphic image (isomorphic to a quadric in the rectilinear case) and conversely every such section satisfying a certain convexity condition gives rise to a Finsler structure on S^n sharing the same geodesics. Using Kodaira deformation theory this allows Bryant to determine the generality of such Finsler structures sharing the same unparametrised geodesics.

Although being related, the results in this article were arrived at independently.

2 2-Segre structures and a quasiholomorphic fibre bundle

2.1 Definitions and examples

Write $W = (\mathbb{R}^m)^*, W' = \mathbb{R}^n$ for $m, n \geq 2$. A vector $v \in W \otimes W'$ is called *decomposable* or *simple* if there exists $w \in W$ and $w' \in W'$ such that $v =$

$w \otimes w'$. We thus may think of the nonzero simple vectors in $W \otimes W'$ as linear maps $\mathbb{R}^m \to \mathbb{R}^n$ of rank one. A linear map between vector spaces which sends simple vectors to simple vectors will itself be called *decomposable* or *simple*. The simple maps in $\mathrm{GL}(W \otimes W')$ are a closed subgroup which will be denoted by $G(m,n)$. Let $x = (x_\alpha)$ be linear coordinates on W and $y = (y^k)$ on W'. Writing $z^k_\alpha = x_\alpha \otimes y^k$, the set of simple vectors in $W \otimes W'$ is given by the homogeneous quadratic equations

$$z^i_\alpha z^k_\beta - z^i_\beta z^k_\alpha = 0 \tag{2.1}$$

and thus is a cone. A subset $\mathcal{S}$ in a real vector space V is called an (m,n)-*Segre cone*, if there exists a linear isomorphism $V \simeq W \otimes W'$ which yields a bijection between $\mathcal{S}$ and the cone of nonzero simple vectors in $W \otimes W'$.

Definition. An (m,n)-*Segre structure* $\mathcal{S}$ on a smooth mn-manifold M is a choice of an (m,n)-Segre cone $\mathcal{S}_p$ in each tangent space of M which varies smoothly from point to point. A manifold equipped with a Segre structure is called a *Segre manifold*.

Alternatively, a Segre structure $\mathcal{S}$ can be described as a first order G-structure with structure group $G = G(m,n)$. Choose standard bases (w^α) for $W = (\mathbb{R}^m)^*$, (w_k) for $W' = \mathbb{R}^n$ and think of $G(m,n)$ as a Lie subgroup of $\mathrm{GL}(mn,\mathbb{R})$ using the linear isomorphism $W \otimes W' \simeq \mathbb{R}^{mn}$

$$w^\alpha \otimes w_i \mapsto e_{\alpha+m(i-1)}, \tag{2.2}$$

where $e = (e_l)$ denotes the standard basis of $\mathbb{R}^{mn}$.

For a smooth vector bundle $E \to M$ of rank n, let $F^*E \to M$ denote the principal right $\mathrm{GL}(n,\mathbb{R})$-*coframe bundle* whose fibre at $p \in M$ consists of the linear isomorphisms $f : E_p \to \mathbb{R}^n$. The Lie group $\mathrm{GL}(n,\mathbb{R})$ acts on $F^*E \to M$ from the right via $(f,g) \mapsto g^{-1} \circ f$.

A coframe $f \in \mathcal{F}^* = F^*TM$ at $p \in M$ is said to be *adapted* to the Segre structure $\mathcal{S}$ if f maps the Segre cone $\mathcal{S}_p \subset T_pM$ onto the Segre cone of simple vectors in $\mathbb{R}^{mn}$ obtained by the identification $\mathbb{R}^{mn} \simeq W \otimes W'$. The group $G(m,n)$ acts simply transitively on the set of adapted coframes at $p \in M$ and thus $\mathcal{S}$ gives rise to a reduction $\pi : F_{\mathcal{S}} \to M$ of the coframe bundle $\lambda : \mathcal{F}^* \to M$ with structure group $G(m,n)$. Conversely it can be shown that every reduction of the coframe bundle with structure group $G(m,n)$ is the reduction of a unique Segre structure $\mathcal{S}$ on M.

Suppose there exist smooth vector bundles $E \to M$ and $E' \to M$ of rank m, n respectively and a smooth vector bundle isomorphism

$$TM \simeq E^* \otimes E',$$

where $E^* \to M$ denotes the dual bundle of $E \to M$. Then, in particular, M inherits an (m,n)-Segre structure $\mathcal{S}$ with associated $G(m,n)$-bundle $\pi : F_{\mathcal{S}} \to M$. In addition, the direct product $P = F^*E \times_M F^*E'$ of the coframe bundles of E and E' yields a principal bundle $\mu : P \to M$ with structure group $\mathrm{GL}(m,\mathbb{R}) \times \mathrm{GL}(n,\mathbb{R})$. The bundle $P \to M$ together with the natural projection

$$\nu : P \to F_{\mathcal{S}}, \quad (f, f') \mapsto {}^t f^{-1} \otimes f'$$

is an equivariant lift of $\pi : F_{\mathcal{S}} \to M$ with respect to the Lie group homomorphism

$$\rho : \mathrm{GL}(m,\mathbb{R}) \times \mathrm{GL}(n,\mathbb{R}) \to G(m,n), \quad (a,b) \mapsto {}^t a^{-1} \otimes b,$$

where the superscript ^{t} denotes the transpose (or dual) linear map. By equivariant lift we mean that $\mu = \pi \circ \nu$ and $\nu(p \cdot g) = \nu(p) \cdot \rho(g)$ for all $p \in P$ and $g \in \mathrm{GL}(m,\mathbb{R}) \times \mathrm{GL}(n,\mathbb{R})$. Conversely such an equivariant lift $P \to M$ of a Segre structure $\pi : F_{\mathcal{S}} \to M$ yields two vector bundles $E \to M$ and $E' \to M$ together with an isomorphism $TM \simeq E^* \otimes E'$ inducing the given Segre structure.

The choice of an isomorphism

$$\Lambda^m E^* \simeq \Lambda^n E'$$

between the determinant line bundles of E^* and E' corresponds to a reduction $R \to M$ of the bundle $P \to M$ with structure group

$$\mathrm{S}(\mathrm{GL}(m,\mathbb{R}) \times \mathrm{GL}(n,\mathbb{R}))$$

consisting of pairs $(a,b) \in \mathrm{GL}(m,\mathbb{R}) \times \mathrm{GL}(n,R)$ which satisfy $\det a \det b = 1$. The reduction $R \to M$ is again an equivariant lift of $\pi : F_{\mathcal{S}} \to M$ with respect to the restriction of ρ to the subgroup $\mathrm{S}(\mathrm{GL}(m,\mathbb{R}) \times \mathrm{GL}(n,\mathbb{R})) \subset \mathrm{GL}(m,\mathbb{R}) \times \mathrm{GL}(n,\mathbb{R})$ and the natural projection $R \to M$. Again, such an equivariant lift $R \to M$ of a given Segre structure $\pi : F_{\mathcal{S}} \to M$ yields two vector bundles $E \to M$ and $E' \to M$ together with isomorphisms $\Lambda^m E^* \simeq \Lambda^n E'$ and $TM \simeq E^* \otimes E'$ the later of which induces the given Segre structure.

Remark. Note that if $n + m$ is odd, the restriction of ρ to the subgroup $S(\mathrm{GL}(m,\mathbb{R}) \times \mathrm{GL}(n,\mathbb{R}))$ gives a Lie group isomorphism

$$\mathrm{S}(\mathrm{GL}(m,\mathbb{R}) \times \mathrm{GL}(n,\mathbb{R})) \simeq G(m,n)$$

and thus in this case a Segre structure always admits an equivariant lift $R \to M$ with structure group $\mathrm{S}(\mathrm{GL}(m,\mathbb{R}) \times \mathrm{GL}(n,\mathbb{R}))$.

Let $\mathfrak{gl}(mn, \mathbb{R})$ denote the Lie algebra of $\mathrm{GL}(mn, \mathbb{R})$. A linear connection $\theta : T\mathcal{F}^* \to \mathfrak{gl}(mn, \mathbb{R})$ is called adapted to $\mathcal{S}$ if θ pulls-back to $F_{\mathcal{S}} \subset \mathcal{F}^*$ to become a principal $G(m,n)$-connection. The Segre structure $\mathcal{S}$ is called *torsion-free*, if it admits an adapted linear connection θ with vanishing torsion, i.e. the torsion-form T satisfies

$$T = \mathrm{d}\eta + \theta \wedge \eta = 0,$$

where η denotes the canonical $\mathbb{R}^{mn}$-valued 1-form on $\mathcal{F}^*$

$$\eta_f(\xi) = f(\lambda'(\xi)), \quad f \in \mathcal{F}^*,\ \xi \in T_f\mathcal{F}^*.$$

A linear subspace $\Pi \subset \mathcal{S}$ is called *simple*. The simple m-dimensional linear subspaces of the form $\Pi \simeq W \otimes \mathbb{R}\{w'\}$ for some vector $w' \in W'$ are called *α-planes*. The simple n-dimensional linear subspaces of the form $\Pi \simeq \mathbb{R}\{w\} \otimes W'$ for some dual vector $w \in W$ are called *β-planes*. An immersed connected manifold[1] $\Sigma \to M$ for which $T_p\Sigma$ is a β-plane for every point $p \in \Sigma$ is called a *proto β-surface*. If, in addition, $\Sigma \to M$ is maximal in the sense of inclusion, then Σ is called a *β-surface*. A Segre structure $\mathcal{S}$ is called *β-integrable* if every β-plane Π is tangent to a unique β-surface $\Sigma \to M$. The notion of a (proto) α-surface and α-integrability are defined analogously. The necessary and sufficient conditions for a Segre structure of type (m,n) to be α- or β-integrable were given in [2, 20] (see also [4] for the complex case). In [19], the case $m = n = 2$ was studied and it was observed that if in the addition to β-integrability one requests that every β-surface is an embedded 2-sphere (or every β-surface an embedded real projective plane), then M must be homeomorphic to $G_2^+(\mathbb{R}^4)$ (or $G_2(\mathbb{R}^4)$).

Note that the group $G(m,n)$ acts from the left on $\mathbb{P}(W)$ by $(a \otimes b) \cdot [w] \mapsto [a(w)]$ and thus yields an associated $\mathbb{RP}^{m-1}$-bundle $F_{\mathcal{S}} \times_{G(m,n)} \mathbb{P}(W) \to M$ whose total space will be denoted by B. The bundle B can be identified with the set of β-planes on M. The natural action of $G(m,n)$ on $\mathbb{P}(W')$ given by $(a \otimes b) \cdot [w'] \mapsto [b(w')]$ yields an $\mathbb{RP}^{n-1}$-bundle $A \to M$ whose total space can be identified with the set of α-planes on M.

Example. The projective linear group $P = \mathrm{PL}(n+m, \mathbb{R})$ acts transitively from the left on the Grassmannian $G_m(\mathbb{R}^{n+m})$ of m-planes in $\mathbb{R}^{n+m}$ by

$$[g] \cdot \Pi = \mathbb{R}\{gv_1, \dots, gv_m\}$$

[1]By an immersed manifold $\Sigma \to M$ we mean an equivalence class of immersions $f : \Sigma \to M$ where $f_1 \sim f_2$ iff $f_2 = f_1 \circ \phi$ for some diffeomorphism $\phi \in \mathrm{Diff}(\Sigma)$.

where $v_1, \ldots, v_m$ is a basis of Π. The stabiliser subgroup of any m-plane $\Pi \in G_m(\mathbb{R}^{n+m})$ may be identified with the closed subgroup $S \subset H$ consisting of elements of the form

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

where $a \in \mathrm{GL}(m, \mathbb{R}), c \in \mathrm{GL}(n, \mathbb{R})$ and $b \in M_{\mathbb{R}}(m, n)$ is a real $(m \times n)$-matrix. Consequently $G_m(\mathbb{R}^{n+m})$ is diffeomorphic to the quotient P/S. Let $\mathfrak{p}$ and $\mathfrak{s}$ denote the Lie-algebras of P and S and $\mathrm{Ad}_{\mathfrak{p}/\mathfrak{s}} : S \to \mathrm{GL}(\mathfrak{p}/\mathfrak{s})$ the adjoint representation of S on the quotient $\mathfrak{p}/\mathfrak{s}$. It can be shown (see for instance [26] for details of this general construction) that the coframe bundle of the Grassmannian $G_m(\mathbb{R}^{n+m}) \simeq P/S$ reduces to a subbundle $\pi_0 : F_0 \to G_m(\mathbb{R}^{n+m})$ whose structure group G is isomorphic to the image of $\mathrm{Ad}_{\mathfrak{p}/\mathfrak{s}}$ and whose total space is diffeomorphic to the quotient P/N where N consists of elements of the form

$$\begin{bmatrix} \mathrm{I}_m & b \\ 0 & \mathrm{I}_n \end{bmatrix}.$$

Here I_k denotes the $(k \times k)$-unit matrix. In the case discussed here, G is easily seen to be $G(m, n)$ and thus the Grassmannian $G_m(\mathbb{R}^{n+m})$ carries an (m, n)-Segre structure $\mathcal{S}_0$. Of course $\mathrm{SL}(n+m, \mathbb{R})$ acts transitively on $G_m(\mathbb{R}^{n+m})$ as well. The quotient of $\mathrm{SL}(n+m, \mathbb{R})$ by the subgroup $\tilde{N}$ consisting of matrices of the form

$$\begin{pmatrix} \mathrm{I}_m & b \\ 0 & \mathrm{I}_n \end{pmatrix}$$

can be identified with the total space of a right principal $\mathrm{S}(\mathrm{GL}(m, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R}))$ bundle $R \to G_m(\mathbb{R}^{n+m})$ which together with the natural map $\nu : R \to F_0$ is an equivariant lift of $\pi_0 : F_0 \to G_m(\mathbb{R}^{n+m})$.

Example. A split-signature metric g on a 4-manifold M is characterised by having components

$$\begin{pmatrix} \mathrm{I}_2 & 0 \\ 0 & -\mathrm{I}_2 \end{pmatrix}$$

in a suitably chosen basis for every tangent space. As in the case of (4,0)-signature a split-signature metric is called self-dual if its Weyl curvature tensor, considered as a bundle-valued 2-form, is its own Hodge-star. Let $[g]$ denote the associated conformal structure $[g] = \{ fg \,|\, f \in C^\infty(M, \mathbb{R}^*) \}$. Locally there exist 1-forms ε^i_α such that a representative g of $[g]$ may be written as

$$g = \varepsilon^1_1 \odot \varepsilon^2_2 - \varepsilon^1_2 \odot \varepsilon^2_1 \tag{2.3}$$

where $\odot$ denotes the symmetric tensor product. A vector $v \in TM$ is called *null* if $g(v,v) = 0$. It follows with (2.3) that the null vectors give rise to a smooth family of (2,2) Segre cones and conversely every such family gives rise to a conformal structure of split-signature on M.

2.2 An almost complex structure

We will henceforth restrict to the case $m = 2$. A $(2,n)$-Segre structure will simply be called a 2-Segre structure. Let $\mathcal{S}$ be a 2-Segre structure on the $2n$-manifold M and $\pi : F_{\mathcal{S}} \to M$ its bundle of adapted coframes with structure group $G = G(2,n)$. Let $\mathfrak{g} \subset \mathfrak{gl}(2n,\mathbb{R})$ denote the Lie algebra of the structure group $G \subset \mathrm{GL}(2n,\mathbb{R})$. For computational purposes it is convenient to introduce the matrices

$$a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad b_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and write e^i_k for the $(n \times n)$-matrix whose entry is 1 at the position (k,i) and 0 otherwise. Using this notation a $\mathfrak{g}$-basis is given by

$$a \otimes \mathrm{I}_n, \quad b_1 \otimes \mathrm{I}_n, \quad b_2 \otimes \mathrm{I}_n, \quad \mathrm{I}_2 \otimes e^i_k.$$

With respect to this basis an $\mathcal{S}$-adapted connection form θ may be written as

$$\theta = \chi \otimes \mathrm{I}_n + \mathrm{I}_2 \otimes \phi \tag{2.4}$$

with $\chi = \omega a + 2\xi_1 b_1 + 2\xi_2 b_2$ and $\phi = \phi^i_k e^k_i$ for some linearly independent 1-forms $\omega, \xi_1, \xi_2, \phi^i_k$ on $F_{\mathcal{S}}$. In terms of the entries θ^j_l, $j, l = 1, \ldots, 2n$, of the connection form θ we have

$$\omega = \theta^2_1, \quad \xi_1 = \frac{1}{2}\left(\theta^1_2 + \theta^2_1\right), \quad \xi_2 = \frac{1}{2}\left(\theta^2_2 - \theta^1_1\right), \quad \phi^i_k = \theta^{2i-1}_{2k-1}, \tag{2.5}$$

for $i, k = 1, \ldots, n$. Write $\xi = \xi_1 + \mathrm{i}\xi_2$ and $\theta = (\omega, \xi, \phi)$. Identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ by

$$(x_1, y_1, \ldots, x_n, y_n) \mapsto (x_1 + \mathrm{i}y_1, \ldots, x_n + \mathrm{i}y_n), \tag{2.6}$$

denote by $\zeta = (\zeta^k)$ the complex-valued 1-form obtained from η via this identification and by $J^{2n}_0 : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ the induced complex structure map. Let $\mathrm{GL}(n,\mathbb{C}) \subset \mathrm{GL}(2n,\mathbb{R})$ denote the closed subgroup of linear maps commuting with J^{2n}_0. Intersecting G with $\mathrm{GL}(n,\mathbb{C})$ gives a closed subgroup $H \subset G$, where $H \subset \mathrm{GL}(n,\mathbb{C})$ is generated by $\mathrm{GL}(n,\mathbb{R})$ and the central subgroup $S^1 \subset \mathrm{GL}(n,\mathbb{C})$ consisting of scalar multiplication by unit

complex numbers. We will henceforth write $H = S^1 \cdot \mathrm{GL}(n,\mathbb{R})$ and denote its elements by $e^{\mathrm{i}\varphi} \cdot b$.

Equip the quotient

$$X_{\mathcal{S}} = F_{\mathcal{S}}/H$$

with its canonical smooth structure so that the quotient projection $\nu : F_{\mathcal{S}} \to X_{\mathcal{S}}$ is a smooth surjective submersion. Note that the 1-forms η^k are ν-semibasic since they are π-semibasic. Moreover since $\theta = (\omega, \xi, \phi)$ is a principal G-connection, it follows with (2.4) and (2.5) that ξ is ν-semibasic as well. Therefore the forms η^k together with ξ_1 and ξ_2 span the ν-semibasic 1-forms on $F_{\mathcal{S}}$.

Lemma 1. *Let $\pi : F_{\mathcal{S}} \to M$ be a 2-Segre structure and $\theta = (\omega, \xi, \phi)$ an adapted connection. Then there exists a unique almost complex structure $\mathfrak{J}$ on $X_{\mathcal{S}}$ such that a complex-valued 1-form on $X_{\mathcal{S}}$ is a (1,0)-form for $\mathfrak{J}$ if and only if its ν-pullback is a linear combination of $\left\{\zeta^1, \ldots, \zeta^n, \xi\right\}$ with coefficients in $C^\infty(F_{\mathcal{S}}, \mathbb{C})$.*

Proof. Denote by

$$\frac{\partial}{\partial \eta^l}, \frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2}, \frac{\partial}{\partial \omega}, \frac{\partial}{\partial \phi^i_k},$$

the vector fields dual to the coframing $(\eta^l, \xi_1, \xi_2, \omega, \phi^i_k)$. Define the map $\tilde{\mathfrak{J}} : TF_{\mathcal{S}} \to TX_{\mathcal{S}}$ by

$$\tilde{\mathfrak{J}}(v) = \nu' \left(-\eta^{2k}(v) \frac{\partial}{\partial \eta^{2k-1}} + \eta^{2k-1}(v) \frac{\partial}{\partial \eta^{2k}} - \xi_2(v) \frac{\partial}{\partial \xi_1} + \xi_1(v) \frac{\partial}{\partial \xi_2} \right).$$

The 1-forms η^l, ξ_1, ξ_2 are ν-semibasic and thus we have $\tilde{\mathfrak{J}}(v + w) = \tilde{\mathfrak{J}}(v)$ for every $v \in TF_{\mathcal{S}}$ and $w \in TF_{\mathcal{S}}$ tangent to the ν-fibres. Since θ is a principal G-connection the equivariance $(R_g)^*\theta = g^{-1}\theta g$ for $g \in G$ together with a short computation gives

$$\left(R_{e^{\mathrm{i}\varphi}\cdot b}\right)^* \xi = e^{-2\mathrm{i}\varphi}\xi, \tag{2.7}$$

for $e^{\mathrm{i}\varphi} \cdot b \in S^1 \cdot \mathrm{GL}(n,\mathbb{R})$. Moreover we have

$$(R_g)^*\eta = g^{-1}\eta \tag{2.8}$$

for every $g \in G$. Using the fact that $S^1 \cdot \mathrm{GL}(n,\mathbb{R}) \subset \mathrm{GL}(n,\mathbb{C})$ together with (2.7) and (2.8) implies

$$\tilde{\mathfrak{J}} \circ (R_{e^{\mathrm{i}\alpha}\cdot b})' = \tilde{\mathfrak{J}}.$$

In other words there exists an almost complex structure $\mathfrak{J} : TX_{\mathcal{S}} \to TX_{\mathcal{S}}$ such that $\tilde{\mathfrak{J}} = \mathfrak{J} \circ \nu'$. Clearly $\mathfrak{J}$ has the desired properties and these properties uniquely characterise $\mathfrak{J}$. $\square$

It is natural to ask when two $\mathcal{S}$-adapted connections induce the same almost complex structure. We have

Lemma 2. *The $\mathcal{S}$-adapted connections $\theta = (\phi, \omega, \xi)$ and $\theta' = (\phi', \omega', \xi')$ induce the same almost complex structure on $X_{\mathcal{S}}$ if and only if $\xi - \xi' = \lambda_k \zeta^k$ for some smooth functions $\lambda_k : F_{\mathcal{S}} \to \mathbb{C}$. In particular any two $\mathcal{S}$-adapted connections with the same torsion induce the same almost complex structure.*

Proof. Let $\mathfrak{J}_\theta$, $\mathfrak{J}_{\theta'}$ denote the almost complex structures with respect to the connections θ, θ' and suppose $\xi' = \xi + \lambda_k \zeta^k$ for some smooth functions $\lambda_k : F_{\mathcal{S}} \to \mathbb{C}$. Let α be a (1,0)-form for $\mathfrak{J}_\theta$. Then we may write

$$\nu^* \alpha = a_k \zeta^k + a\xi = a_k \zeta^k + a\left(\xi' - \lambda_k \zeta^k\right) = \left(a_k - \lambda_k\right)\zeta^k + a\xi'$$

for some smooth functions $a, a_k : F_{\mathcal{S}} \to \mathbb{C}$, thus showing that α is a (1,0)-form for $\mathfrak{J}_{\theta'}$. Conversely suppose $\mathfrak{J}_\theta = \mathfrak{J}_{\theta'}$. Note that $\xi - \xi'$ is π-semibasic and may thus be written as

$$\xi - \xi' = \lambda_k \zeta^k + \lambda'_k \bar{\zeta}^k$$

for some smooth functions $\lambda_k, \lambda'_k : F_{\mathcal{S}} \to \mathbb{C}$. Let α be a (1,0)-form for $\mathfrak{J}_\theta$. Then we may write

$$\nu^* \alpha = a_k \zeta^k + a\xi = a'_k \zeta^k + a'\xi' = a'_k \zeta^k + a'\left(\xi - \lambda_k \zeta^k - \lambda'_k \bar{\zeta}^k\right)$$

for some smooth functions $a, a', a_k, a'_k : F_{\mathcal{S}} \to \mathbb{C}$. Thus it follows

$$(a - a')\xi + \left(a_k - a'_k + a'\lambda_k\right)\zeta^k + a'\lambda'_k \bar{\zeta}^k = 0$$

which can hold for an arbitrary (1,0)-form α if and only if $\lambda'_k = 0$. Finally it is easy to check that if (ω, ξ, ϕ) and (ω', ξ', ϕ') are two $\mathcal{S}$-adapted connections with the same torsion, then there exist n smooth complex-valued functions a_k on $F_{\mathcal{S}}$ such that

$$\begin{aligned} \omega' - \omega = \mathrm{Re}(a_k)\,\mathrm{Im}(\zeta^k), \quad \xi' - \xi = \frac{1}{2\mathrm{i}}\bar{a}_k \zeta^k, \\ \left(\phi'\right)^i_l - \phi^i_l = \mathrm{Re}(a_l \zeta^i) + \delta^i_l\,\mathrm{Re}(a_k)\,\mathrm{Re}(\zeta^k). \end{aligned} \tag{2.9}$$

□

2.3 Integrability conditions

Denote by $\mathcal{A}^{1,0}_{\mathcal{S}}$ and $\mathcal{A}^{0,1}_{\mathcal{S}}$ the complex-valued π-semibasic 1-forms on $F_{\mathcal{S}}$ which can be written as $a_k\zeta^k$ and $a_k\bar{\zeta}^k$ respectively. Here a_k are smooth complex-valued functions on $F_{\mathcal{S}}$. Furthermore set

$$\mathcal{A}^{p,q}_{\mathcal{S}} = \Lambda^p\left(\mathcal{A}^{1,0}_{\mathcal{S}}\right) \otimes \Lambda^q\left(\mathcal{A}^{0,1}_{\mathcal{S}}\right),$$

so that the complex-valued π-semibasic k-forms $\mathcal{A}^k_{\mathcal{S}}$ on $F_{\mathcal{S}}$ decompose as

$$\mathcal{A}^k_{\mathcal{S}} = \bigoplus_{p+q=k} \mathcal{A}^{p,q}_{\mathcal{S}}.$$

Denote by $\tau = (\tau^i)$ the complex-valued torsion-form of the connection $\theta = (\omega, \xi, \phi)$ obtained by the identification $\mathbb{R}^{2n} \simeq \mathbb{C}^n$.

Proposition 1. *The connection forms (ω, ξ, ϕ) of an $\mathcal{S}$-adapted connection θ satisfy*

$$\mathrm{d}\zeta = -\big(\mathrm{i}\,(\omega - \xi)\,\mathrm{I}_n + \phi\big) \wedge \zeta - \mathrm{i}\,\xi\,\mathrm{I}_n \wedge \bar{\zeta} + \tau. \tag{2.10}$$

Proof. Straightforward computation. □

Remark. Here $\bar{\zeta}$ denotes the $\mathbb{C}^n$-valued 1-form on $F_{\mathcal{S}}$ which is obtained by complex conjugation of the entries of ζ.

Define the curvature forms

$$\begin{aligned} \Omega &= \mathrm{d}\omega + \omega \wedge \mathrm{i}\,(\xi - \bar{\xi}), \\ \Xi &= \mathrm{d}\xi + \xi \wedge \mathrm{i}\left(\bar{\xi} - 2\omega\right), \\ \Phi &= \mathrm{d}\phi + \phi \wedge \phi - \omega \wedge \left(\xi + \bar{\xi}\right)\mathrm{I}_n. \end{aligned} \tag{2.11}$$

Differentiating the structure equation (2.10) gives the *Bianchi-identity*

$$\mathrm{d}\tau = \big(\mathrm{i}\,(\Omega - \Xi)\,\mathrm{I}_n + \Phi\big) \wedge \zeta + \mathrm{i}\,\Xi\,\mathrm{I}_n \wedge \bar{\zeta}. \tag{2.12}$$

Proposition 2. *The almost complex structure $\mathfrak{J}$ is integrable if and only if Ξ and the torsion components τ^i lie in $\mathcal{A}^{2,0}_{\mathcal{S}} \oplus \mathcal{A}^{1,1}_{\mathcal{S}}$. In particular for $n \geq 3$ every torsion-free $\mathcal{S}$-adapted connection gives rise to an integrable $\mathfrak{J}$.*

Remark. The integrability conditions for the almost complex structure $\mathfrak{J}$ can be obtained by applying [24, Theorem 4]. We we will instead use Proposition 1.

Proof of Proposition 2. Using the characterisation of $\mathfrak{J}$ provided in Lemma 1, the first statement is an immediate consequence of the structure equation (2.10), the definition of the curvature form Ξ in (2.11) and the Newlander-Nirenberg theorem. In order to prove the second statement we need to show that for $n \geq 3$ the condition $\tau = 0$ implies $\Xi \in \mathcal{A}^{2,0}_{\mathcal{S}} \oplus \mathcal{A}^{1,1}_{\mathcal{S}}$. Since the curvature form Ξ is a π-semibasic 2-form we may write

$$\Xi = x_{kl}\zeta^k \wedge \zeta^l + \tilde{x}_{kl}\bar{\zeta}^k \wedge \zeta^l + \hat{x}_{kl}\bar{\zeta}^k \wedge \bar{\zeta}^l \tag{2.13}$$

for some smooth complex-valued functions $x_{kl}, \tilde{x}_{kl}, \hat{x}_{kl}$ on $F_{\mathcal{S}}$. Writing out the Bianchi-identity (2.12) in components for $\tau = 0$ gives

$$0 = (\mathrm{i}(\Omega - \Xi)\delta^i_k + \Phi^i_k) \wedge \zeta^k + \mathrm{i}\,\Xi \wedge \bar{\zeta}^i,$$

replacing Ξ with the expansion (2.13) we get

$$0 = \cdots + \mathrm{i}\hat{x}_{kl}\bar{\zeta}^k \wedge \bar{\zeta}^l \wedge \bar{\zeta}^i$$

where the unwritten summands do not contain forms in $\mathcal{A}^{0,3}_{\mathcal{S}}$. If $n \geq 3$ there is for every choice of indices k, l an index $i \neq k$, $i \neq l$ so that the Bianchi-identity can hold if and only if $\hat{x}_{kl} = 0$ which is equivalent to Ξ lying in $\mathcal{A}^{2,0}_{\mathcal{S}} \oplus \mathcal{A}^{1,1}_{\mathcal{S}}$. $\square$

Remark. Recall that (2,2)-Segre structures correspond to conformal structures of split-signature and thus are always torsion-free. It is easy to check that the logical value of the curvature condition $\Xi \in \mathcal{A}^{2,0}_{\mathcal{S}} \oplus \mathcal{A}^{1,1}_{\mathcal{S}}$ does not depend on the choice of a particular adapted torsion-free connection, but only on $\mathcal{S}$. We leave it to the reader to check that this curvature condition corresponds to self-duality of the associated conformal 4-manifold of split-signature.

In fact, it can be shown[2] that for $n = 2$ the almost complex structure $\mathfrak{J}$ is integrable if and only if θ is torsion-free and the associated split-signature conformal structure is self-dual. For $n \geq 3$ the almost complex structure $\mathfrak{J}$ is integrable if and only if θ is torsion-free.

2.4 A quasiholomorphic fibre bundle

Suppose $\mathfrak{J}$ is integrable, so that the total space of the bundle $\rho : X_{\mathcal{S}} \to M$ is a complex $(n+1)$-manifold. By construction the ρ-fibres are smoothly embedded submanifolds of $X_{\mathcal{S}}$ diffeomorphic to $\mathrm{GL}(2,\mathbb{R})/\mathrm{GL}(1,\mathbb{C})$. We will argue next, that $(X_{\mathcal{S}}, \mathfrak{J})$ is a quasiholomorphic fibre bundle.

[2] This will be done in detail and more generality elsewhere.

Definition. Let $\pi : F \to M$ be a fibre bundle and $\mathfrak{J}$ an almost complex structure on F. Then $(F, \mathfrak{J})$ is called *quasiholomorphic* if $\mathfrak{J}$ is integrable and the π-fibres are holomorphically embedded complex submanifolds.

Pulling back $\zeta^1, \ldots, \zeta^n$ and ξ with a local section s of $\nu : F_{\mathcal{S}} \to X_{\mathcal{S}}$ gives a local basis for the (1,0)-forms of $X_{\mathcal{S}}$. Since $\zeta^1, \ldots, \zeta^n$ is semibasic for the projection $\pi : F_{\mathcal{S}} \to M$ it follows that $s^*\zeta^i$ is ρ-semibasic. In particular the forms $s^*\zeta^i$ vanish when pulled back to a fibre of ρ and therefore $s^*\xi$ pulls back to the fibres to be a non-vanishing complex-valued 1-form which depends on s only up to complex multiples. It follows that the fibres of $\rho : X_{\mathcal{S}} \to M$ are holomorphically embedded Riemann surfaces with respect to the complex structure induced by ξ. Next we will identify this Riemann surface.

Note that the image of the standard embedding of $\mathbb{RP}^n$ into $\mathbb{CP}^n$ given by

$$[x]_{\mathbb{RP}^n} \mapsto [x]_{\mathbb{CP}^n}$$

consists precisely of those elements $[z]$ for which $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are linearly dependent. We will write $\mathbb{CP}^n \setminus \mathbb{RP}^n$ to denote the complex submanifold of $\mathbb{CP}^n$ obtained by cutting out the image of the $\mathbb{RP}^n$ standard embedding.

Let

$$\alpha = \begin{pmatrix} \alpha^1_1 & \alpha^1_2 \\ \alpha^2_1 & \alpha^2_2 \end{pmatrix}$$

be the Maurer-Cartan form of $\mathrm{GL}(2, \mathbb{R})$. The complex valued 1-form $\hat{\xi} = (\alpha^1_2 + \alpha^2_1) + \mathrm{i}(\alpha^2_2 - \alpha^1_1)$ is semibasic for the quotient projection

$$\tau : \mathrm{GL}(2, \mathbb{R}) \to \mathrm{GL}(2, \mathbb{R})/\mathrm{GL}(1, \mathbb{C})$$

and invariant under the right action of $\mathrm{GL}(1, \mathbb{C})$ up to a complex multiple. It follows that the quotient $Q = \mathrm{GL}(2, \mathbb{R})/\mathrm{GL}(1, \mathbb{C})$ admits a unique structure of a Riemannian surface such that a complex valued 1-form $\mu \in \mathcal{A}^1(Q, \mathbb{C})$ is of type (1,0) if and only if $\tau^*\mu$ is a complex multiple of $\hat{\xi}$. Consider the map $\lambda : \mathrm{GL}(2, \mathbb{R}) \to \mathbb{CP}^1 \setminus \mathbb{RP}^1$ given by

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \begin{bmatrix} a_{11} + \mathrm{i}a_{12} \\ a_{21} + \mathrm{i}a_{22} \end{bmatrix}.$$

Straightforward computations show that λ is a smooth surjection whose fibres are the $\mathrm{GL}(1, \mathbb{C})$ orbits and that λ pulls back the (1,0)-forms of $\mathbb{CP}^1 \setminus \mathbb{RP}^1$ to complex multiples of $\hat{\xi}$, hence Q with its complex structure is biholomorphic to $\mathbb{CP}^1 \setminus \mathbb{RP}^1$ via the biholomorphism $\lambda \circ \tau^{-1} : Q \to \mathbb{CP}^1 \setminus \mathbb{RP}^1$.

Let $\kappa : \mathrm{GL}(2, \mathbb{R}) \to G$ denote the Lie group embedding $a \mapsto {}^t a^{-1} \otimes \mathrm{I}_n$ and let $\iota_f : G \to F_{\mathcal{S}}$ denote the canonical inclusion of G at f. Then for

every $f \in F_{\mathcal{S}}$ the map $\lambda_f = \nu \circ \iota_f \circ \kappa : \mathrm{GL}(2,\mathbb{R}) \to X_{\mathcal{S}}$ is a smooth surjective submersion onto the ρ-fibre at $\pi(f)$ whose fibres are the $\mathrm{GL}(1,\mathbb{C})$ orbits. Hence there exists a unique diffeomorphism φ_f such that the diagram

$$\begin{tikzcd} \mathrm{GL}(2,\mathbb{R}) \arrow[d, "\lambda_f"'] \arrow[dr, "\tau"] & \\ (X_{\mathcal{S}})_{\pi(f)} \arrow[r, "\varphi_f"] & Q \end{tikzcd}$$

commutes. Give the quotient Q the unique complex structure J for which φ_f is a biholomorphism. Since $\theta = (\omega, \xi, \phi)$ is a principal G-connection, λ_f pulls back the (1,0)-forms of the ρ-fibres to complex multiples of $\hat{\xi}$. This shows that τ pulls back the J (1,0)-forms to complex multiples of $\hat{\xi}$. Thus Q and hence the fibres of ρ are biholomorphic to $\mathbb{CP}^1 \setminus \mathbb{RP}^1$.

In [2, 20] it was shown that for $n \geq 3$ a 2-Segre structure is β-integrable if and only if it is torsion-free and for $n = 2$ if and only if it is self-dual. Summarising we have.

Theorem 1. *Let $\pi : F_{\mathcal{S}} \to M$ be a β-integrable 2-Segre structure. Then there exists a canonical almost complex structure $\mathfrak{J}$ on $X_{\mathcal{S}}$ so that $(X_{\mathcal{S}}, \mathfrak{J})$ is a quasiholomorphic fibre bundle with fibre $\mathbb{CP}^1 \setminus \mathbb{RP}^1$.*

Proof. We pick an $\mathcal{S}$-adapted connection without torsion and let $\mathfrak{J}$ be the associated almost complex structure on $X_{\mathcal{S}}$ whose existence is guaranteed by Lemma 1. Then by Proposition 2 and the subsection 2.4, the almost complex structure $\mathfrak{J}$ is integrable and $(X_{\mathcal{S}}, \mathfrak{J})$ is a quasiholomorphic fibre bundle with fibre $\mathbb{CP}^1 \setminus \mathbb{RP}^1$. Finally, by Lemma 2, any other $\mathcal{S}$-adapted torsion-free connection gives rise to the same almost complex structure $\mathfrak{J}$. $\square$

3 Reductions of β-integrable 2-Segre structures

We will henceforth consider the β-integrable case and assume $\rho : X_{\mathcal{S}} \to M$ to be equipped with its canonical integrable almost complex structure $\mathfrak{J}$ with respect to which $(X_{\mathcal{S}}, \mathfrak{J})$ is a quasiholomorphic fibre bundle. By construction the sections of the bundle $\rho : X_{\mathcal{S}} \to M$ correspond to reductions of the principal G-bundle $\pi : F_{\mathcal{S}} \to M$ with structure group $S^1 \cdot \mathrm{GL}(n,\mathbb{R})$. In this section we will show that the torsion-free $S^1 \cdot \mathrm{GL}(n,\mathbb{R})$-reductions of $F_{\mathcal{S}}$ are in one-to-one correspondence with the sections $\sigma : M \to X_{\mathcal{S}}$ having holomorphic image $\sigma(M) \subset X_{\mathcal{S}}$. This will done using exterior differential systems

(EDS). The notation and terminology for EDS is chosen to be consistent with [6].

3.1 The structure equations of the reduction

A basis for the Lie algebra of $S^1 \cdot \mathrm{GL}(n,\mathbb{R}) \subset \mathrm{GL}(2n,\mathbb{R})$ is given by

$$a \otimes \mathrm{I}_n, \quad \mathrm{I}_2 \otimes e^i_k.$$

Suppose $R \to M$ is a torsion-free $S^1 \cdot \mathrm{GL}(n,\mathbb{R})$-structure with adapted connection θ. Write

$$\theta = a\alpha \otimes \mathrm{I}_n + \mathrm{I}_2 \otimes \beta,$$

for some 1-form α and some $\mathfrak{gl}(n,\mathbb{R})$-valued 1-form β on R. Let ζ denote the pullback of the canonical $\mathbb{C}^n$-valued 1-form to R, then ζ satisfies

$$\mathrm{d}\zeta = -\left(\mathrm{i}\alpha\, \mathrm{I}_n + \beta\right) \wedge \zeta, \tag{3.1}$$

as was already observed in [8].

3.2 Reductions having holomorphic image

Let (X,J) be a complex manifold and $f : M \to X$ a real even dimensional submanifold. Then (f,M) is called a *complex submanifold of* X if M admits the (unique) structure of a complex manifold such that f is a holomorphic map. Clearly if (f,M) is a complex submanifold, then the image $f(M) \subset X$ admits the unique structure of a complex manifold, such that the inclusion into X is a holomorphic map. In light of this we will also say that the complex submanifold $f : M \to X$ has *holomorphic image*.

Lemma 3. *Let (X,J) be a complex $(n+1)$-manifold, $(\mu^1,\dots,\mu^n,\kappa) \in \mathcal{A}^1(X,\mathbb{C})$ a basis for the (1,0)-forms of J and $f : \Sigma \to X$ a $2n$-submanifold with*

$$f^*\left(\mathrm{i}\mu^1 \wedge \bar{\mu}^1 \wedge \cdots \wedge \mathrm{i}\mu^n \wedge \bar{\mu}^n\right) \neq 0. \tag{3.2}$$

Then (f,Σ) is a complex submanifold if and only if

$$f^*\left(\kappa \wedge \mu^1 \wedge \cdots \wedge \mu^n\right) = 0.$$

Moreover through every point $p \in X$ passes such a complex submanifold.

Proof. The non-degeneracy (3.2) implies that

$$f^*\left(\kappa \wedge \mu^1 \wedge \cdots \wedge \mu^n\right) = 0$$

if and only if there exist smooth complex-valued functions x_i on Σ such that

$$f^*\kappa = x_i f^*\mu^i. \tag{3.3}$$

Suppose $f : \Sigma \to X$ is a complex submanifold with respect to the complex structure $\tilde{J}$ on Σ. Then the holomorphicity of f and the non-degeneracy (3.2) imply that the forms $f^*\mu^i$ for $i = 1, \dots, n$ are a basis for the (1,0)-forms of $\tilde{J}$. It follows that f pulls back κ to a (1,0)-form of $\tilde{J}$, i.e. there exist smooth complex-valued functions x_i on Σ such that (3.3) holds. Conversely suppose there are smooth complex-valued functions x_i on Σ such that (3.3) holds. Let $\tilde{J} : T\Sigma \to T\Sigma$ be the almost complex structure for which the forms $f^*\mu^i$ for $i = 1, \dots, n$ span the (1,0)-forms. Then (3.3) implies that $\tilde{J}$ is integrable and f a holomorphic map.

The second statement is an easy consequence of the existence of holomorphic coordinates on X and its proof is thus omitted. □

On $F_{\mathcal{S}}$ define the exterior differential system

$$\mathcal{I} = \langle \xi \wedge \zeta^1 \wedge \cdots \wedge \zeta^n \rangle$$

together with the independence condition

$$Z = \mathrm{i}\zeta^1 \wedge \bar{\zeta}^1 \wedge \cdots \wedge \mathrm{i}\zeta^n \wedge \bar{\zeta}^n.$$

The EDS $(\mathcal{I}, Z)$ is of interest because of the following

Lemma 4. *Let $\sigma : M \to X_{\mathcal{S}}$ be an $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$-reduction of $F_{\mathcal{S}}$ and $\tilde{\sigma} : U \to F_{\mathcal{S}}$ a local coframing covering σ. Then $\tilde{\sigma}$ is an integral manifold of $(\mathcal{I}, Z)$ if and only if $\sigma|_U : U \to X_{\mathcal{S}}$ is a complex submanifold.*

Proof. Let $s : \rho^{-1}(U) \to F_{\mathcal{S}}$ be a local section of the bundle $\nu : F_{\mathcal{S}} \to X_{\mathcal{S}}$ and let $\mu^i = s^*\zeta^i$ for $i = 1, \dots, n$ and $\kappa = s^*\xi$ be a local basis for the (1,0)-forms on $\rho^{-1}(U)$. Then

$$\begin{aligned} \nu^*\mu^i &= (s \circ \nu)^*\zeta^i = (R_t)^*\zeta^i, \\ \nu^*\kappa &= (s \circ \nu)^*\xi = (R_t)^*\xi, \end{aligned}$$

for some smooth function $t : \pi^{-1}(U) \to S^1 \cdot \mathrm{GL}(n, \mathbb{R})$. Recall that for $e^{i\varphi} \cdot b \in S^1 \cdot \mathrm{GL}(n, \mathbb{R})$ we have

$$\begin{aligned} \left(R_{e^{\mathrm{i}\varphi} \cdot b}\right)^*\xi &= e^{-2\mathrm{i}\varphi}\xi, \\ \left(R_{e^{\mathrm{i}\varphi} \cdot b}\right)^*\zeta &= \left(e^{-\mathrm{i}\varphi} \cdot b^{-1}\right)\zeta. \end{aligned}$$

This yields

$$\nu^*\left(\mathrm{i}\mu^1\wedge\bar\mu^1\wedge\cdots\wedge\mathrm{i}\mu^n\wedge\bar\mu^n\right)=(\det b)^{-2}\,Z\neq 0$$

for some smooth map $b:\pi^{-1}(U)\to \mathrm{GL}(n,\mathbb{R})$ and

$$\nu^*\kappa=e^{-2\mathrm{i}\varphi}\xi$$

for some smooth function $\varphi:\pi^{-1}(U)\to\mathbb{R}$. Hence we get

$$(\sigma|_U)^*\left(\mathrm{i}\mu^1\wedge\bar\mu^1\wedge\cdots\wedge\mathrm{i}\mu^n\wedge\bar\mu^n\right)=\left((\det b)^{-2}\circ\tilde\sigma\right)\tilde\sigma^*Z$$

which vanishes nowhere since $\tilde\sigma$ is a π-section. Therefore according to Lemma 3, $\sigma|_U:U\to X_{\mathcal{S}}$ is a complex submanifold if and only if

$$(\sigma|_U)^*\left(\kappa\wedge\mu^1\wedge\cdots\wedge\mu^n\right)=\left(\left(\frac{e^{-(n+2)\mathrm{i}\varphi}}{\det b}\right)\circ\tilde\sigma\right)\tilde\sigma^*\left(\xi\wedge\zeta^1\wedge\cdots\wedge\zeta^n\right)=0.$$

□

Recall that $S^1\cdot\mathrm{GL}(n,\mathbb{R})\subset\mathrm{GL}(n,\mathbb{C})$ and we can thus look for reductions $\sigma:M\to X_{\mathcal{S}}$ whose associated almost complex structure $\mathfrak{J}_\sigma$ is integrable.

Proposition 3. *Let $\sigma:M\to X_{\mathcal{S}}$ be an $S^1\cdot\mathrm{GL}(n,\mathbb{R})$-reduction of $\pi:F_{\mathcal{S}}\to M$. Then the following two statements are equivalent:*

(i) The almost complex structure $\mathfrak{J}_\sigma$ is integrable.

(ii) Any local coframing $\tilde\sigma:U\to F_{\mathcal{S}}$ covering σ is an integral manifold of $(\mathcal{I},Z)$.

Proof. Since $\tilde\sigma$ is a π-section we have $\tilde\sigma^*Z\neq 0$. Write $\chi^i=\tilde\sigma^*\zeta^i$. The local coframing $\tilde\sigma$ is adapted to the $S^1\cdot\mathrm{GL}(n,\mathbb{R})$-reduction σ and thus the forms χ^i are a local basis of the (1,0)-forms of $\mathfrak{J}_\sigma$. By Newlander-Nirenberg $\mathfrak{J}_\sigma$ is integrable if and only if there exist complex-valued 1-forms π^i_k such that

$$d\chi^i=\pi^i_k\wedge\chi^k.$$

Using the structure equation (2.10) we get

$$\mathrm{d}\chi^i=\tilde\sigma^*d\zeta^i=\tilde\pi^i_k\wedge\chi^k-\mathrm{i}\tilde\sigma^*\xi\wedge\bar\chi^i \tag{3.4}$$

for some complex-valued 1-forms $\tilde\pi^i_k$. Write

$$\tilde\sigma^*\xi=x_k\chi^k+y_k\bar\chi^k$$

for some smooth complex-valued functions x_k,y_k on U. Then (3.4) implies that $\mathfrak{J}_\sigma$ is integrable on U if and only if the functions y_k all vanish. This condition is equivalent to $\tilde\sigma$ being an integral manifold of $(\mathcal{I},Z)$. □

We are now ready to prove the main statement.

Theorem 2. *Let $\pi : F_{\mathcal{S}} \to M$ be a β-integrable 2-Segre structure. Then an $S^1 \cdot \mathrm{GL}(n,\mathbb{R})$-reduction $R \subset F_{\mathcal{S}}$ is torsion-free if and only if $\nu(R) \subset X_{\mathcal{S}}$ is a complex submanifold.*

Proof. Let $\nu(R) = \sigma(M)$ for some ρ-section $\sigma : M \to X_{\mathcal{S}}$ which has holomorphic image, then by Lemma 4 and Proposition 3, the almost complex structure $\mathfrak{J}_\sigma$ is integrable. This is equivalent to ξ satisfying $\xi = x_k\zeta^k$ for some smooth complex-valued functions x_k on R. Pulling back the structure equation (2.10) to $R \subset F_{\mathcal{S}}$ gives

$$\mathrm{d}\zeta = -\left(\mathrm{i}\left(\omega - x_k\zeta^k\right)\mathrm{I}_n + \phi\right) \wedge \zeta - \mathrm{i}\, x_k\zeta^k\,\mathrm{I}_n \wedge \bar{\zeta}. \tag{3.5}$$

Define

$$\begin{aligned}\alpha &= \omega - \mathrm{Im}(x_k)\,\mathrm{Im}(\zeta^k),\\ \beta^j_l &= \phi^j_l - \mathrm{Re}(\mathrm{i}\bar{x}_l\zeta^j) - \delta^j_l\,\mathrm{Im}(x_k)\,\mathrm{Re}(\zeta^k),\end{aligned}$$

then the 1-form $\theta = a\alpha \otimes \mathrm{I}_n + \mathrm{I}_2 \otimes \beta$ is a linear connection on R which satisfies

$$\mathrm{d}\zeta = -\left(\mathrm{i}\alpha\mathrm{I}_n + \beta\right) \wedge \zeta, \tag{3.6}$$

thus R is torsion-free. Conversely suppose the reduction $\sigma : M \to X_{\mathcal{S}}$ is torsion-free, so that on $R = (\nu^{-1}\circ\sigma)(M)$ there exists a linear connection $\theta = a\alpha \otimes \mathrm{I}_n + \mathrm{I}_2 \otimes \beta$ satisfying (3.6). Pulling back (ω, ξ, ϕ) to R gives

$$\begin{aligned}\omega &= \alpha + a_k\left(\zeta^k + \bar{\zeta}^k\right) + \mathrm{i}\tilde{a}_k\left(\zeta^k - \bar{\zeta}^k\right)\\ \xi &= x_k\zeta^k + \tilde{x}_k\bar{\zeta}^k\\ \phi^i_k &= \beta^i_k + f^i_{kl}\left(\zeta^k + \bar{\zeta}^k\right) + \mathrm{i}\tilde{f}^i_{kl}\left(\zeta^k - \bar{\zeta}^k\right)\end{aligned} \tag{3.7}$$

for some smooth complex-valued functions $a_k, \tilde{a}_k, x_k, \tilde{x}_k, f^i_{kl}, \tilde{f}^i_{kl}$ on R. Subtracting (3.5) from (3.6) and using (3.7) gives in components

$$0 = \cdots + \mathrm{i}\left(x_k\zeta^k + \tilde{x}_k\bar{\zeta}^k\right) \wedge \bar{\zeta}^i \tag{3.8}$$

where the unwritten summands are not of the form $\bar{\zeta}^k \wedge \bar{\zeta}^i$. It follows that (3.8) can hold for every $i = 1, \ldots, n$ if and only if $\tilde{x}_k = 0$. $\square$

Corollary 1. *Locally every β-integrable 2-Segre structure $\pi : F_{\mathcal{S}} \to M$ can be reduced to a torsion-free $S^1 \cdot \mathrm{GL}(n,\mathbb{R})$-structure.*

Proof. For a given point $p \in M$, choose $q \in X_{\mathcal{S}}$ with $\rho(q) = p$ and a coordinate neighbourhood U_p. Let μ^i, $i = 1, \ldots, n$ and κ be a basis for the (1,0)-forms on $\rho^{-1}(U_p)$ as constructed in Lemma 4. Using Lemma 3 there exists a complex $2n$-submanifold $f : \Sigma \to \rho^{-1}(U_p)$ passing through q for which

$$f^* \left(\mathrm{i}\mu^1 \wedge \bar{\mu}^1 \wedge \cdots \wedge \mathrm{i}\mu^n \wedge \bar{\mu}^n \right) \neq 0.$$

Since the $\pi : F_{\mathcal{S}} \to M$ pullback of a volume form on M is a nowhere vanishing multiple of $Z = \mathrm{i}\mu^1 \wedge \bar{\mu}^1 \wedge \cdots \wedge \mathrm{i}\mu^n \wedge \bar{\mu}^n$, the ρ pullback of a volume form on U_p is a nowhere vanishing multiple of Z and hence $\rho \circ f : \Sigma \to U_p$ is a local diffeomorphism. Composing f with the locally available inverse of this local diffeomorphism one gets a local ρ-section which is defined in a neighbourhood of p and which is a complex submanifold. Applying Theorem 2 it follows that $\pi : F_{\mathcal{S}} \to M$ locally has an underlying torsion-free $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$-structure. $\square$

4 The flat case

In this section we apply the obtained results to the Grassmannian of oriented 2-planes in $\mathbb{R}^{n+2}$ which carries a torsion-free 2-Segre together with an adapted connection of vanishing curvature and which in addition carries an orientation.

Definition. A 2-Segre structure $\pi : F_{\mathcal{S}} \to M$ is called *2-orientable* if the structure group can be reduced to $G^+(2, n) = \mathrm{GL}^+(W) \otimes \mathrm{GL}(W')$. A 2-Segre structure whose structure group has been reduced to $G^+(2, n)$ is called *2-oriented.*

Remark. In the case where the 2-Segre structure is induced by a decomposition $TM \simeq E^* \otimes E'$, the reduction of the structure group to $G^+(2, n)$ consists of choosing an orientation on the vector bundle $E \to M$. For n odd, a 2-Segre structure $\pi : F_{\mathcal{S}} \to M$ is 2-orientable if and only if M is orientable. For $n = 2k$, a manifold carrying a 2-Segre structure is always orientable since $G(2, n) \subset \mathrm{GL}^+(4k, \mathbb{R})$.

The arguments used to prove Theorem 1 also show the following: If $\mathcal{S}$ is a β-integrable 2-oriented 2-Segre structure on M with principal $G^+ = G^+(2, n)$-bundle $\pi : F_{\mathcal{S}} \to M$ of adapted coframes. Then $\rho : X_{\mathcal{S}} = F_{\mathcal{S}} / (S^1 \cdot \mathrm{GL}(n, \mathbb{R})) \to M$ together with its canonical almost complex structure $\mathfrak{J}$ is a quasiholomorphic fibre bundle with fibre $\mathrm{GL}^+(2, \mathbb{R}) / \mathrm{GL}(1, \mathbb{C}) \simeq D^2$, the open unit disk in $\mathbb{C}$.

4.1 The Grassmannian of oriented 2-planes

Of course, the projective linear group $\mathrm{PL}(n+2,\mathbb{R})$ also acts transitively from the left on the Grassmannian $G_2^+(\mathbb{R}^{n+2})$ of oriented 2-planes in $\mathbb{R}^{n+2}$ and the stabiliser subgroup of any element $\Pi \in G_2^+(\mathbb{R}^{n+2})$ may be identified with the subgroup S consisting of elements of the form

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

where $a \in \mathrm{GL}^+(2,\mathbb{R}), c \in \mathrm{GL}(n,\mathbb{R})$ and $b \in M_{\mathbb{R}}(2,n)$ is a real $(2\times n)$-matrix. Let $\mu : \mathrm{PL}(n+2,\mathbb{R}) \to G_2^+(\mathbb{R}^{n+2}) \simeq \mathrm{PL}(n+2,\mathbb{R})/S$ be the quotient projection and write

$$\tilde{\theta} = \begin{pmatrix} \alpha & \beta \\ \eta & \gamma \end{pmatrix}$$

for the Maurer-Cartan form of $\mathrm{PL}(n+2,\mathbb{R})$. The real matrix-valued 1-forms α,β,γ,η have sizes according to the block decomposition of the Lie group S and satisfy $\mathrm{Tr}(\alpha)+\mathrm{Tr}(\gamma)=0$. Let H^i_k denote the vector fields dual to the forms η with respect to the coframing $\tilde{\theta}$. Let $N \subset \mathrm{PL}(n+2,\mathbb{R})$ be the closed normal subgroup given by

$$N = \left\{ \begin{bmatrix} \mathrm{I}_2 & b \\ 0 & \mathrm{I}_n \end{bmatrix} \,\middle|\, b \in M_{\mathbb{R}}(2,n) \right\}$$

whose elements will be denoted by $[b]$. The quotient Lie group S/N is isomorphic to $G^+(2,n)$ and thus $\mathrm{PL}(n+2,\mathbb{R})/N$ is the total space of a right principal $G^+(2,n)$-bundle over $G_2^+(\mathbb{R}^{n+2})$. Consider the smooth map

$$\begin{aligned} \varphi^i_k : \mathrm{PL}(n+2,\mathbb{R}) &\to TG_2^+(\mathbb{R}^{n+2}) \\ p &\mapsto \mu'_p(H^i_k(p)) \end{aligned}$$

The Maurer-Cartan equation $\mathrm{d}\tilde{\theta} + \tilde{\theta}\wedge\tilde{\theta} = 0$ implies that the form η is basic for the quotient projection $\mathrm{PL}(n+2,\mathbb{R}) \to \mathrm{PL}(n+2,\mathbb{R})/N$. Therefore the maps φ^i_k are invariant under the right action of N and thus descend to smooth maps $\mathrm{PL}(n+2,\mathbb{R})/N \to TG_2^+(\mathbb{R}^{n+2})$. The images $\varphi^i_k(p)$ for a given point $p \in \mathrm{PL}(n+2,\mathbb{R})$ are linearly independent and thus induce a map φ into the coframe bundle of $G_2^+(\mathbb{R}^{n+2})$. The maps φ^i_k can be arranged so that the induced map φ from $\mathrm{PL}(n+2,\mathbb{R})/N$ into the coframe bundle of $G_2^+(\mathbb{R}^{n+2})$ pulls back the components of the canonical $\mathbb{C}^n$-valued 1-form to

$$\mathrm{i}\left(\eta^k_1 + \mathrm{i}\eta^k_2\right).$$

It follows again with the Maurer-Cartan equation that φ embeds $\mathrm{PL}(n+2,\mathbb{R})/N$ as a smooth right principal $G^+(2,n)$-subbundle of the coframe bundle of $G_2^+(\mathbb{R}^{n+2})$. This subbundle will be denoted by $\pi_0 : F_0 \to G_2^+(\mathbb{R}^{n+2})$ and the projection $\mathrm{PL}(n+2,\mathbb{R}) \to F_0$ by υ. Write

$$\begin{aligned}
\tilde{\omega} &= \alpha_1^2, \\
2\,\tilde{\xi} &= \left(\alpha_2^1 + \alpha_1^2\right) + \mathrm{i}\left(\alpha_2^2 - \alpha_1^1\right), \\
\tilde{\phi} &= \gamma - \mathrm{I}_n \alpha_2^2, \\
\zeta^i &= \mathrm{i}(\eta_1^i + \mathrm{i}\eta_2^i).
\end{aligned}$$

Then straightforward computations show that the forms $(\tilde{\omega}, \tilde{\xi}, \tilde{\phi})$ transform under the right action of $G^+(2,n)$ as the connection forms of an $\mathcal{S}_0$-adapted connection do. Moreover we have

$$\mathrm{d}\zeta = -\left(\mathrm{i}\left(\tilde{\omega} - \tilde{\xi}\right)\mathrm{I}_n + \tilde{\phi}\right) \wedge \zeta - \mathrm{i}\tilde{\xi}\mathrm{I}_n \wedge \bar{\zeta},$$

and

$$\begin{aligned}
(R_{[b]})^* \tilde{\omega} &= \tilde{\omega} + \mathrm{Re}(a_k)\,\mathrm{Im}(\zeta^k), \\
(R_{[b]})^* \tilde{\xi} &= \tilde{\xi} + \frac{1}{2\mathrm{i}} \bar{a}_k \zeta^k, \\
(R_{[b]})^* \tilde{\phi}_l^i &= \tilde{\phi}_l^i + \mathrm{Re}(a_l \zeta^i) + \delta_l^i\,\mathrm{Re}(a_k)\,\mathrm{Re}(\zeta^k),
\end{aligned}$$

where we have written $a_k = -\mathrm{i}\,(b_{1k} + \mathrm{i}b_{2k})$. This implies that there exists an adapted torsion-free connection (ω, ξ, ϕ) on F_0 such that

$$\upsilon^*(\omega, \xi, \phi) = (\tilde{\omega}, \tilde{\xi}, \tilde{\phi}), \ \bmod \eta, \tag{4.1}$$

i.e. (4.1) holds up to linear combinations of the elements of η. Furthermore the Maurer-Cartan equation implies that the curvature forms of this connection all vanish. Summarising we have proved the

Proposition 4. *The Grassmannian of oriented 2-planes admits a 2-oriented 2-Segre structure $\mathcal{S}_0$ together with an adapted, torsion-free, flat connection (ω, ξ, ϕ) such that $\upsilon^*(\omega, \xi, \phi) = (\tilde{\omega}, \tilde{\xi}, \tilde{\phi})$, $\bmod\, \eta$, holds.*

Let $\rho : X_0 \to G_2^+(\mathbb{R}^{n+2})$ be the D^2-bundle associated to the 2-oriented torsion-free 2-Segre structure $\pi_0 : F_0 \to G_2^+(\mathbb{R}^{n+2})$ and $\mathfrak{J}_0$ its canonical almost complex structure which makes $(X_0, \mathfrak{J}_0)$ into a quasiholomorphic fibre bundle with fibre D^2. Its total space X_0 can be identified with the quotient $\mathrm{PL}(n+2,\mathbb{R})/\tilde{H}$ where $\tilde{H}$ is the closed Lie subgroup

$$\tilde{H} = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \;\middle|\; a \in \mathrm{GL}(1,\mathbb{C}), b \in M_{\mathbb{R}}(2,n), c \in \mathrm{GL}(n,\mathbb{R}) \right\}.$$

Write an element $[g] \in \mathrm{PL}(n+2,\mathbb{R})$ as $[g_1,\ldots,g_{n+2}]$ where the elements g_k are column-vectors well defined up to a common non-zero factor. Consider the smooth map

$$\begin{aligned}\lambda : \mathrm{PL}(n+2,\mathbb{R}) &\to \mathbb{CP}^{n+1} \setminus \mathbb{RP}^{n+1}\\ [g_1,g_2,\ldots,g_{n+2}] &\mapsto [g_1 + \mathrm{i}g_2].\end{aligned}$$

Clearly λ is a surjective submersion whose fibres are the $\tilde{H}$-orbits and thus induces a diffeomorphism $\varphi : X_0 \to \mathbb{CP}^{n+1} \setminus \mathbb{RP}^{n+1}$. Therefore $\rho_0 = \rho \circ \varphi^{-1}$: $\mathbb{CP}^{n+1} \setminus \mathbb{RP}^{n+1} \to G_2^+(\mathbb{R}^{n+2})$ is a bundle with fibre D^2. Explicitly ρ_0 is given by

$$[z] \mapsto \mathbb{R}\{\mathrm{Re}(z), \mathrm{Im}(z)\}$$

and the 2-plane $\mathbb{R}\{\mathrm{Re}(z), \mathrm{Im}(z)\}$ is oriented by declaring $\mathrm{Re}(z), \mathrm{Im}(z)$ to be a positively oriented basis.

Proposition 5. *There exists a biholomorphic fibre bundle isomorphism*

$$\varphi : (X_0, \mathfrak{J}_0) \to \mathbb{CP}^{n+1} \setminus \mathbb{RP}^{n+1}$$

covering the identity on $G_2^+(\mathbb{R}^{n+2})$.

Proof. Using Lemma 1 and Proposition 4 it sufficient to show that λ pulls-back the (1,0)-forms of $\mathbb{CP}^{n+1} \setminus \mathbb{RP}^{n+1}$ to linear combinations of the forms

$$\zeta^1, \ldots, \zeta^n, \tilde{\xi}.$$

This is a computation which causes no difficulties and so we omit it. □

4.2 Smooth quadrics without real points

If V is a real vector space, $V_{\mathbb{C}} = V \otimes \mathbb{C}$ will denote its complexification and $\mathbb{P}(V_{\mathbb{C}}) = (V_{\mathbb{C}} \setminus \{0\})/\mathbb{C}^*$ its complex projectivisation. An element $[z] \in \mathbb{P}(V_{\mathbb{C}})$ for which z is a simple vector is called a *real point*.

The aim of this subsection is to show that the smooth quadrics $Q \subset \mathbb{CP}^{n+1} = \mathbb{P}(\mathbb{R}^{n+2}_{\mathbb{C}})$ without real points are in one-to-one correspondence with the sections of the bundle $\rho_0 : \mathbb{CP}^{n+1} \setminus \mathbb{RP}^{n+1} \to G_2^+(\mathbb{R}^{n+2})$ having holomorphic image. This is done by reducing the problem to the case $n = 1$ which was shown to be true in [22, Corollary 2] (see also [7, Theorem 9]).

Let $\Pi \subset \mathbb{R}^{n+2}$ be a 3-dimensional linear subspace. Choosing an isomorphism $\mathbb{R}^3 \simeq \Pi$ induces an embedding of the 2-sphere $S^2 \simeq G_2^+(\mathbb{R}^3) \hookrightarrow G_2^+(\mathbb{R}^{n+2})$. Clearly the image of this embedding and its induced smooth

structure do not depend on the chosen isomorphism and thus Π determines a smoothly embedded 2-sphere in $G_2^+(\mathbb{R}^{n+2})$ which will be denoted by S_Π. Moreover the isomorphism $\mathbb{R}^3 \simeq \Pi$ induces a holomorphic embedding $\mathbb{CP}^2 \simeq \mathbb{P}(\Pi_{\mathbb{C}}) \hookrightarrow \mathbb{CP}^{n+1}$ and thus an embedding $\mathbb{CP}^2 \setminus \mathbb{RP}^2 \hookrightarrow \mathbb{CP}^{n+1} \setminus \mathbb{RP}^{n+1}$. Again the image of this embedding and its induced complex structure do not depend on the chosen isomorphism and thus Π determines a holomorphically embedded submanifold $Y_\Pi \subset \mathbb{CP}^{n+1} \setminus \mathbb{RP}^{n+1}$. Restricting the base point projection $\rho_0 : \mathbb{CP}^{n+1} \setminus \mathbb{RP}^{n+1} \to G_2^+(\mathbb{R}^{n+2})$ to Y_Π gives a D^2-bundle $\rho_\Pi : Y_\Pi \to S_\Pi$ which is isomorphic to the bundle $\rho_0^2 : \mathbb{CP}^2 \setminus \mathbb{RP}^2 \to G_2^+(\mathbb{R}^3)$, $[z] \mapsto \mathbb{R}\{\mathrm{Re}(z), \mathrm{Im}(z)\}$.

Recall that for a smooth algebraic hypersurface $X \subset \mathbb{P}(V_{\mathbb{C}})$, the Gauss map $\mathcal{G}_X : X \to G_{n-1}(V_{\mathbb{C}})$ sends a point $x \in X$ to the tangent hyperplane of X at x. The dual variety X^* is now defined to be the image of X under the Gauss map. Usually $\mathcal{G}_X$ is assumed to take values in $\mathbb{P}(V_{\mathbb{C}}^*) = \mathbb{P}((V_{\mathbb{C}})^*) \simeq G_{n-1}(V_{\mathbb{C}})$.

Lemma 5. *Let V be a n-dimensional real vector space with $n \geq 2$ and $Q \subset \mathbb{P}(V_{\mathbb{C}})$ a smooth quadric without real points. Then the dual of Q is a smooth quadric without real points in $\mathbb{P}(V_{\mathbb{C}}^*)$.*

Proof (see also Proposition 4 of [7]). Let Q be defined by the zero locus of a non-degenerate quadratic form $K : V_{\mathbb{C}} \to \mathbb{C}$ which does not vanish on real points. Then there exists a basis $v_1, \ldots, v_n$ of V, inducing an isomorphism $\mathbb{P}(V_{\mathbb{C}}) \simeq \mathbb{CP}^{n-1}$, so that

$$K\left(\sum_{k=1}^{n} v_k \otimes z_k\right) = \sum_{k=1}^{n} e^{\mathrm{i}p_k}(z_k)^2 \tag{4.2}$$

for some real numbers $p_1, \ldots, p_n$ which satisfy

$$0 = p_1 \leq p_2 \leq \cdots \leq p_n < \pi.$$

Conversely it is easy to check that each quadratic form of type (4.2) defines a smooth quadric without real points in $\mathbb{P}(V_{\mathbb{C}})$. A tangent hyperplane to Q at $[z_1, \ldots, z_n]^t \in \mathbb{CP}^{n-1}$ may be identified with the point

$$\left[\bar{z}_1 e^{-\mathrm{i}p_1}, \ldots, \bar{z}_n e^{-\mathrm{i}p_n}\right]^t \in \left(\mathbb{CP}^{n-1}\right)^*.$$

Denoting a point in $(\mathbb{CP}^{n-1})^*$ by $[w_1, \ldots, w_n]^t$, the set of tangent hyperplanes to Q is given by the solution of the equation

$$\sum_{k=1}^{n} e^{\mathrm{i}p_k}(w_k)^2 = 0,$$

which defines a smooth quadric without real points in $(\mathbb{CP}^{n-1})^*$. $\square$

It follows with Lemma 5 that the intersection of a smooth quadric without real points with a real k-plane gives again a smooth quadric without real points. More precisely we have the following

Lemma 6. *Let V be a n-dimensional $\mathbb{R}$-vector space with $n \geq 2$ and $\Pi \subset V$ a k-dimensional linear subspace with $2 \leq k \leq n$. Suppose $Q \subset \mathbb{P}(V_{\mathbb{C}})$ is a smooth quadric without real points, then $Q_\Pi = Q \cap \mathbb{P}(\Pi_{\mathbb{C}})$ is a smooth quadric without real points in $\mathbb{P}(\Pi_{\mathbb{C}})$.*

Proof. We prove the lemma by downwards induction on k. Suppose $k = n$, then there is nothing to show. Now suppose Lemma 6 holds true for some fixed k. Let $Q \subset \mathbb{P}(V_{\mathbb{C}})$ be a smooth quadric without real points, $\Pi \subset V$ a $(k-1)$-dimensional linear subspace and $\tilde{\Pi} \subset V$ a k-dimensional linear subspace containing Π. Now by assumption $Q_{\tilde{\Pi}} = Q \cap \mathbb{P}(\tilde{\Pi}_{\mathbb{C}})$ is a smooth quadric without real points in $\mathbb{P}(\tilde{\Pi}_{\mathbb{C}})$. The intersection of the real hyperplane $\mathbb{P}(\Pi_{\mathbb{C}})$ with $Q_{\tilde{\Pi}}$ can only be singular if $\mathbb{P}(\Pi_{\mathbb{C}})$ is tangent to $Q_{\tilde{\Pi}}$ at some point $q \in Q_{\tilde{\Pi}}$. This is not possible according to Lemma 5. It follows that $Q_\Pi = \mathbb{P}(\Pi_{\mathbb{C}}) \cap Q_{\tilde{\Pi}}$ is a smooth algebraic hypersurface in $\mathbb{P}(\Pi_{\mathbb{C}})$ and thus a quadric. The hypersurface Q_Π cannot contain real points since $Q_{\tilde{\Pi}}$ does not contain real points. $\square$

Using Lemma 6 we are ready to prove the

Theorem 3. *The sections of the D^2-bundle $\rho_0 : \mathbb{CP}^{n+1} \backslash \mathbb{RP}^{n+1} \to G_2^+(\mathbb{R}^{n+2})$ having holomorphic image are in one-to-one correspondence with the smooth quadrics $Q \subset \mathbb{CP}^{n+1}$ without real points.*

Proof. Let $\sigma : G_2^+(\mathbb{R}^{n+2}) \to \mathbb{CP}^{n+1} \backslash \mathbb{RP}^{n+1}$ be a ρ_0-section with holomorphic image $Q = \operatorname{im} \sigma$. Let $\Pi \subset \mathbb{R}^{n+2}$ be a 3-dimensional linear subspace and $\iota_\Pi : S_\Pi \to G_2^+(\mathbb{R}^{n+2})$, $\tilde{\iota}_\Pi : Y_\Pi \to \mathbb{CP}^{n+1} \setminus \mathbb{RP}^{n+1}$ the corresponding embedded submanifolds. Then the map $\sigma \circ \iota_\Pi : S_\Pi \to \mathbb{CP}^{n+1} \setminus \mathbb{RP}^{n+1}$ is smooth and takes values in Y_Π. Consequently the induced map $\sigma_\Pi : S_\Pi \to Y_\Pi$ is an injective immersion and thus, since S_Π is compact, a smooth embedding. Set

$$Q_\Pi = Q \cap Y_\Pi = \sigma_\Pi(S_\Pi),$$

then $Q_\Pi \subset Y_\Pi$ is a smoothly embedded submanifold. Now Chow's theorem implies that Q is a smooth algebraic hypersurface. Suppose $P : \mathbb{C}^{n+2} \to \mathbb{C}$ is a homogeneous polynomial defining Q and let $P_\Pi : \mathbb{C}^3 \to \mathbb{C}$ denote the homogeneous polynomial obtained by pulling back P to $\Pi_{\mathbb{C}} \simeq \mathbb{C}^3$. The

map P_Π is a homogeneous polynomial of the same degree as P which has no real points, since P has no real points. Under the identification $Y_\Pi \simeq \mathbb{CP}^2 \setminus \mathbb{RP}^2$, Q_Π becomes a smoothly embedded submanifold of $\mathbb{CP}^2 \setminus \mathbb{RP}^2$ defined by the zero locus of the homogeneous polynomial P_Π. Since Q_Π is diffeomorphic to the 2-sphere, the genus of Q_Π is 0 and thus by the degree-genus formula for smooth plane algebraic curves

$$g = \frac{1}{2}(d-1)(d-2),$$

the degree of P_Π must be 1 or 2. However since Q_Π has no real points the degree of P_Π and thus the degree of P must be 2.

Conversely let $Q \subset \mathbb{CP}^{n+1}$ be a smooth quadric without real points. Let $\{\Pi^\iota\}_{\iota\in I}$ be a family of 3-dimensional linear subspaces of $\mathbb{R}^{n+2}$ so that the submanifolds S_{Π^ι} cover $G_2^+(\mathbb{R}^{n+2})$. Let Q_{Π^ι} denote the intersection of Q with $\mathbb{P}(\Pi^\iota_{\mathbb{C}})$ which by Lemma 6 is a smooth quadric without real points. According to [22, Corollary 2] each such quadric is the image of a unique section $\sigma_\iota : S_{\Pi^\iota} \to X_{\Pi^\iota}$. Now for any two $\Pi^{\iota_1}, \Pi^{\iota_2}$ the spheres $S_{\Pi^{\iota_1}}$ and $S_{\Pi^{\iota_2}}$ are either disjoint or intersect in exactly two points. Since for a given Q_{Π^ι} the section σ_ι is unique, it follows that Q_{Π^ι} intersects each ρ_{Π^ι}-fibre in exactly one point. This implies that the sections σ_{ι_1} and σ_{ι_2} agree on intersection points and thus the family $\{\sigma_\iota\}_{\iota\in I}$ gives rise to a unique global section $\sigma : G_2^+(\mathbb{R}^{n+2}) \to \mathbb{CP}^{n+1} \setminus \mathbb{RP}^{n+1}$ with image Q. □

Finally we get the

Corollary 2. *The torsion-free $S^1 \cdot \mathrm{GL}(n, \mathbb{R})$-reductions $R \subset F_0$ are in one-to-one correspondence with the smooth quadrics $Q \subset \mathbb{CP}^{n+1}$ without real points.*

Proof. This follows immediately from Theorems 2 and 3. □

Remark. For $n = 2$, the case of conformal 4-manifolds of split-signature, Corollary 2 can also be deduced by applying the results in [19]. One could also look for $S^1 \cdot \mathrm{GL}(2, \mathbb{R})$-reductions whose associated almost complex structure is not only integrable, but for which the corresponding conformal structure $[g]$ also contains a Kähler-metric. This, and the related problem in (4,0)-signature has been studied in [12] (see also [19, Theorem D]). Moreover for $n = 2$, Theorem 2 has an analogue in (4,0)-signature due to Salamon [25].

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Thomas Mettler
Mathematical Sciences Research Institute
17 Gauss Way
Berkeley, CA-94720, USA
tmettler@msri.org